# On decoherence in quantum algorithm via dynamic models for quantum measurement

C.P Sun and H.Zhan
*Institute of Theoretical Physics, Academia Sinica,Beijing 100080, China*

X.F.Liu
*Department of Mathematics, Peking University, Beijing 100871, China*
(February 1, 2008)



The possible effect of environment on the efficiency of a quantum algorithm is considered explicitely. It is illustrated through the example of Shor's prime factorization algorithm that this effect may be disastrous. The influence of environment on quantum computation is probed on the basis of its analogy to the problem of wave function collapse in quantum measurement.Techniques from the Hepp-Colemen approach and its generalization are used to deal with decoherence problems in quantum computation including dynamic mechanism of decoherence , quantum error avoiding tricks and calculation of decoherence time.

**PACS number(s)**: 03.65-BZ, 71.10.Ad,89.70.+c

## 1. Introduction

Quantum computations (QC) can be understood as a quantum- mechanical time evolution of certain quantum systems (so-called qubits) [1-9], in which the nonclassical dynamic feature, such as the quantum coherence of states, plays a dominant role. Indeed, it is the purely quantum characters that makes it possible for a theoretical quantum computer to solve certain hard mathematical problems efficiently. In this respect, perhaps the most important example is Shor's prime factorization algorithm [6]. As the quantum computation is a quantum process, preserving coherence, at least to some extent, throughout the whole process is thus an essential requirement. In fact, the decoherence resulting from the coupling with environment may make an quantum algorithm invalid and may cause unwelcome exponential increase of errors in output results [10-12]. Actually, the decoherence process was even regarded as a mechanism for enforcing classical behavior in the macroscopic realm [13]. In this view a decohered quantum computer becomes a classical one. To overcome this difficulty caused by decoherence, some schemes have been proposed in the last several years [14-20]. Among them are the quantum error correcting technique inspired by the classical error correction theory and the schemes avoiding decoherence presented in connection with the strategy preventing decoherence from certain subsets of quantum states [21-23]. The latter was systematically described in the framework of error avoiding quantum coding (EAQC). For the mathematical details we refer the readers to ref.[23].

It is recongnized that there exists a substantially close relation between the problem of decoherence in quantum computation and the problem of wave function collapse(WFC, also called von Neumann's reduction) in quantum measurement [24]. In the view point of quantum dynamics [25-35] for quantum measurement, both the measured system and the measuring instrument (detector) obey the Schrodinger equation and the dynamic evolution governed by their interaction can result in the WFC under certain conditions. For example, the collapse happens if the detector contains a great number of particles or if the detector is in a state with a very large quantum number.These two cases are usually referred to as macroscopic limit and classical limit respectively [30]. We recall that in the traditional theory of quantum measurement [24], the WFC postulate is only an extra assumption added to the ordinary quantum mechanics. Under this postulate, once we measure an observable and obtain a definite value $a_k$ the state of the system must collapse into the corresponding eigenstate $|k\rangle$ from a coherent superposition $|\phi\rangle = \sum_k c_k|k\rangle\langle k|$. In the terminology of density matrix this process can be described by a projection $\rho = |\phi\rangle\langle\phi| \to \hat{\rho} = \sum |c_k|^2|k\rangle\langle k|$ from a pure state to a mixed state. This projection process means the loss of quantum coherence. There is a strong resemblance between this process and the quantum decoherence of a quantum computer resulting from the coupling with the surrounding environment.Thus, with the correspondence of the environment surrounding the quantum computer to the measurement instrument monitoring the measured system, some obtained results in the quantum dynamic models [26-36,] based on Hepp-Colemen (HC) approach [25] for quantum measurement can be applied to discuss decoherence problems in quantum computation such as the dynamic mechanism of decoherence in quantum computer, the quantum error avoiding techniques and the calculation of the decoherence time of a quantum computation process. We can also re-consider the strategy grouping the quantum states of qubits to form decoherence free subsets [21-23] and analyse the dynamic process decohering the qubits beyond the decoherence free subsets .

In comparison with the dynamic theory of quantum measurement, the system of qubits in quantum computation is an open system $S$ (similar to the measured system) surrounded by an environment ( similar to the measuring instrument,detector). The environment may

cause two unwelcome effects on computation process, namely,causing the states of the qubit system to continuously decohere to approach classical states [13, 25-36]and dissipating the energy of the qubit system into the environment. Mathematically, they are respectively described by vanishing of the off-diagonal and diagonal elements of the reduced density matrix of $S$ . The dissipation effect of imperfect isolation happens at the relaxation time-scale $\tau_{rel}$. It is relatively easy to make systems having a very large $\tau_{rel}$ and thus allowing a reasonable number of operations to complete [10]. In contrast, the effect of decoherence, is much more insidious [10, 37-42] because the coherence information leaks out into the environment in a time scale $\tau_d$ much shorter than $\tau_{rel}$ as a quantum system evolves [10, 37-42]. In fact, it is rather difficult to realize a quantum system with the time scale $\tau_d$ for decoherence smaller than that for the dissipation. For example, let us consider an oscillator in a superposition of two coherent states separated by a distance $l$ from each other. Suppose they interact linearly with a bath of oscillators at temperature T. Then the decoherence time for this system is linearly proportional to the relaxation time with a rather small ratio $(\frac{\lambda_{them}}{l})^2$ at high temperatures where $\lambda_{them}$ is the thermal de Broglie wavelength [10, 37-42]. This implies that the qubits decohere much faster than they dissipate. Thus, the sensibility of quantum computation mainly depends on $\tau_d$rather than $\tau_{rel}$. For this reason , the present discussions in this paper only focus on the decoherence problem.

The arrangement of this paper is as follows. In sec. 2, we consider the influence of environment on Shor's prime factoring algorithm; From sec. 3 to sec.7 we consider decoherence problem caused by environment with concrete models; In sec. 9, we discuss the universality of environment in the weakly coupling limit; Finally in sec. 8 we draw some conclusions.

## 2. Decoherence in Shor Factoring Algorithm

In this section we illustrate the possible influence of environment on the validity of a quantum algorithm through an example, Shor's prime factorization algorithm . In this case, we recall, the so called quantum computer has two registers in use. According to Shor's method [6], to factor a number $n$ one should first of all choose a number $x$ . Then the first step is to put the first register in the uniform superposition of q states $|a\rangle(a = 0, 1, ....q-1)$ and the second one in a single state $|0\rangle$. This leaves the machine in the state

$$|\phi(0)\rangle = \frac{1}{\sqrt{q}}\sum_{a=0}^{q-1}|a\rangle \otimes |0\rangle \tag{2.1}$$

Next, one computes $x^q mod.(n)$ in the second register, leaving the machine in the state

$$|\phi(t)\rangle = \frac{1}{\sqrt{q}}\sum_{a=0}^{q-1}|a\rangle \otimes |x^a mod.(n)\rangle$$

$$\equiv \frac{1}{\sqrt{q}}\sum_{a=0}^{q-1}|a, x^a mod.(n)\rangle \tag{2.2}$$

Then one performs a Fourier transform $A_q$ on the first register. This leaves the machine in the state

$$|\phi_F(t)\rangle = \frac{1}{q}\sum_{a=0}^{q-1}\sum_{c=0}^{q-1}\exp[\frac{2\pi iac}{q}]|c, x^a mod.(n)\rangle \tag{2.3}$$

Finally one observes the machine. One easily finds that the probability that the machine ends in a particular state $|c, x^k mod.(n)\rangle \equiv |c, k(x; n)\rangle$ is

$$p(c, k) = \frac{1}{q^2}|\sum_{a:x^a=x^k mod.(n)}^{q-1}\exp[\frac{2\pi iac}{q}]|^2 \tag{2.4}$$

Shor shows that if $c$ lies in a particular region one can determine a nontrival factor of $n$ from the value of $c$. Denote the one-try-success probability of this method by $p_s$. Then one has the following result:

$$p_s \geq r\phi(r)p(c, k) \geq 1/\ln n \geq 1/3\ln n \tag{2.5}$$

where $r$ is the least integer such that $x^r \equiv 1(mod.n)$ and $\phi$ is Euler's totient function.

Now let us take the influence of environment into account to some extent. Asume that the environment is comprised by $N$ particles. In this case we denote by $\phi'(0), \phi'_F(t), p'(c, k)$ and $p'_s$ the correspondences of $\phi(0), \phi_F(t), p(c, k)$ and $p_s$ respectively. Then we have

$$|\phi'(0)\rangle = \frac{1}{\sqrt{q}}\sum_{a=0}^{q-1}|a\rangle \otimes |0\rangle \otimes |e\rangle \tag{2.5}$$

where $|e\rangle = |e_1\rangle \otimes |e_2\rangle \otimes ... \otimes |e_N\rangle$ is the initial state of the environment without correlation with the state of the machine. Here, $|e_k\rangle(k = 1, 2, ..., N)$ denotes the initial states of individual particle comprising the environment . Accordingly,

$$|\phi'(t)\rangle = \frac{1}{\sqrt{q}}\sum_{a=0}^{q-1}|a\rangle \otimes |x^a mod.(n)\rangle \otimes |e[a]\rangle$$

$$\equiv \frac{1}{\sqrt{q}}\sum_{a=0}^{q-1}|a, x^a mod.(n)\rangle \otimes |e[a]\rangle \tag{2.6}$$

where $|e[a]\rangle = U_a(t)|e\rangle$ and $U_a(t)$ is the effective evolution operator of the environment correlated with the state $|a\rangle$. For simplicity we do not consider the influence

of environment in the process of the Fourier transform $A_q$.Thus we have

$$|\phi'_F(t)\rangle = \frac{1}{q}\sum_{a=0}^{q-1}\sum_{c=0}^{q-1}\exp[\frac{2\pi iac}{q}]|c, x^a mod.(n)\rangle \otimes |e[a]\rangle$$

As the only difference between the present model and the original one is the involvment of the environment variables $|e[a]\rangle$ in the entanglement, to proceed along with the discussion we should consider the reduced density matrix

$$\rho(t) = Tr_e(|\phi_F(t)\rangle\langle\phi_F(t)|) =$$

$$\frac{1}{q^2}\sum_{a=0}^{q-1}\sum_{c=0}^{q-1}\sum_{a'=0}^{q-1}\sum_{c'=0}^{q-1}\exp[\frac{2\pi i(ac-a'c')}{q}]\times$$

$$\langle e[a']|e[a]\rangle|c, x^a mod.(n)\rangle\langle c', x^{a'} mod.(n)| \tag{2.8}$$

Here we have traced over the environment variables. We notice that the contribution of environment is given by the transition matrix element

$$F(a,a') \equiv \langle e[a']|e[a]\rangle = \langle e|U^\dagger_{a'}(t)U_a(t)|e\rangle \tag{2.9}$$

$F(a,a')$ is usually called decohering factor. Now it directly follows that

$$p'(c,k) = Tr\{\rho(t)|c,k\rangle\langle c,k|\}$$

$$= \frac{1}{q^2}\sum_{a,a':x^a=x^k mod.(n)=x^{a'}}^{q-1}\exp[\frac{2\pi i(a-a')c}{q}]F(a,a') \tag{2.10}$$

We are now in a position to consider two extreme cases. For the first case, suppose that the qubit system is completely isolated. In this case we have $U_{a'} = U_a$ for $a' \neq a$, so $\langle e[a']|e[a]\rangle = 1$ . As a result we get

$$p'(c,k) = \frac{1}{q^2}|\sum_{a:x^a=x^k mod.(n)}^{q-1}\exp[\frac{2\pi iac}{q}]|^2$$

$$= p(c,k) \tag{2.11}$$

For the second case, suppose that the environment causes a complete decoherence. If we indexed the elements of the reduced density matrix by $a$ and $a'$, then this means that its off-diagonal elements vanish. Such a case has been formulated in the dynamic theory of quantum measurement as a consequence of a certain factorizable structure of the effective evolution operator of environment[30-36]. In fact, if $U_a$ can be factorized as

$$U_a(t) = \prod_j^N U_a^j(t) \tag{2.12}$$

where $U_a^j(t)$ only concerns the j'th particle in the environment, the decohering factor can be expressed as $N$-multiple product

$$F(a,a') = \prod_j^N \langle e_j|U^{j\dagger}_{a'}(t)U^j_a(t)|e_j\rangle$$

$$\equiv \prod_j^N |F^j(a,a') \tag{2.13}$$

of the decohering factors $F^j(a,a') = \langle e|U^{j\dagger}_{a'}(t)U^j_a(t)|e\rangle$ with norms less than unity. In the macroscopic limit $N \to \infty$, it is possible that $F(a,a') \to 0$, for $a' \neq a$, namely,$\langle e[a']|e[a]\rangle = \delta_{aa'}$. Then we have

$$p'(c,k) = \frac{1}{q^2}[(q-1-k)/r]$$

$$\leq \frac{1}{q^2}\frac{q}{r} = \frac{1}{qr} \tag{2.14}$$

and

$$p'(s) = r\phi(r)p'(c,k) =$$

$$\phi(r)/q \leq \phi(r)/n^2 \leq 1/n \tag{2.15}$$

In the remaining part of this section let us proceed on to discuss the possible influence of environment on the efficiency of Shor's prime factorization algorithm.Generally speaking, a deterministic algorithm is said to be efficient if the number of the computation steps taken to excute it increases no faster than a polynomial function of $\ln N$ where $N$ is the input. For a randomized algorithm this definition should be modified to fit in the probability character. Suppose the one-try-success probability of an randomized algorithm $A$ is $s$, then $A$ is said to be efficient if $\forall\varepsilon > 0, \exists\ p(x)$ such that $\forall N \cdot (1-s)^{p(\ln N)} < \varepsilon$, where $p(x)$ is a polynomial. Obviously, the polynomial $p(x)$ here should have real coefficients and satisfy $p(\ln N) > 0$. All the polynomials appearing in the following are tacitly assumed to have this property. It is also clear that in quantum computations all algorithms should be randomized ones.

Let $A$ be a quantum algorithm. Suppose for an input $N$ the one-try-success probability of $A$ is $f(N)$ where $f$ is

a real continuous function defined on the real line.Then we have the following lemma.

**Lemma**. *If there exists a polynomial $p(x)$ such that*

$$\lim_{N\to\infty}(1-f(N))^{p(\ln N)}<1 \tag{2.16}$$

*then A is efficient. Conversely, if for an arbitrary polynomial $p(x)$ we have*

$$\lim_{N\to\infty}(1-f(N))^{p(\ln N)}\geq 1 \tag{2.17}$$

*A is not efficient.*

**Proof**. Let $p(x)$ be a polynomial such that $\lim_{N\to\infty}(1-f(N))^{p(\ln N)}<1$. Then $\forall\varepsilon>0,\exists\alpha(\varepsilon)$ such that $(\lim_{N\to\infty}(1-f(N))^{p(\ln N)})^{\alpha(\varepsilon)}<\varepsilon$. Namely, $\lim_{N\to\infty}(1-f(N))^{p(\ln N)\alpha(\varepsilon)}<\varepsilon$. Defining

$$p'(x)\equiv\alpha(\varepsilon)p(x)$$

we come to the conclusion that there exists some $N_0$ such that $\forall N>N_0$, $(1-f(N))^{p'(\ln N)}<\varepsilon$. It is now evident that one can choose a suitable polynomial $q(x)$ such that $\forall N$, $(1-f(N))^{q(\ln N)}<\varepsilon$. This proves the first part of the lemma.

For the second part of the lemma, if the conclusion were not true, $\forall\varepsilon>0$,there would exist a polynomial $p_\varepsilon(x)$ such that $\forall N$, $(1-f(N))^{p_\varepsilon(\ln N)}<\varepsilon$.Thus we would have $\lim_{N\to\infty}(1-f(N))^{p_\varepsilon(\ln N)}\leq\varepsilon$,leading to the contradiction $1\leq\varepsilon$. The lemma is consequently proved.

Before concluding this section let us take $A$ to be Shor's prime factorization algorithm and return to the above mentioned two extreme cases. For the first case, we have $f(N)>1/3\ln N$. As a result,

$$\lim_{N\to\infty}(1-f(N))^{3\ln N}\leq$$

$$\lim_{N\to\infty}(1-1/(3\ln N))^{3\ln N}$$

$$=1/e<1 \tag{2.17}$$

So according to the lemma $A$ is efficient. For the second case,we have $f(N)\leq 1/N$. It is easy to prove $\lim_{N\to\infty}(1-1/N)^{\ln^m N}=1$ for all ntegers $m$ so for all polynomials $p(x)$ $\lim_{N\to\infty}(1-1/N)^{p(\ln N)}=1$.Consequently, for all polynomials $p(x)$

$$\lim_{N\to\infty}(1-f(N))^{p(\ln N)}$$

$$\geq\lim_{N\to\infty}(1-1/N)^{p(\ln N)}=1 \tag{2.17}$$

This means that the algorithm is no longer efficient in this case.

## 3. Dynamic Model for Decoherence in Quantum Computation: Generalized Hepp-Coleman Approach

In this section we begin to study decoherence problems caused dynamically by environment. Since the influence of environment on quantum computations is reflected in the decohering factor $\langle e[a']|e[a]\rangle$ caused by coupling, as is shown in the last section,our discussion will concern the microscopic dynamics of the interaction between a qubit system and the environment. It is modeled in terms of a generalization of the HC model of the wave function collapse in quantum measurement.

In the generalized HC model [30-32 ], the environment E or the detector is made up of $N$ particles and has the free Hamiltonian in a general form

$$\hat{H}_D=\sum_k^N\hat{H}_k \tag{3.1}$$

where the single particle Hamiltonian $\hat{H}_k$ only depends on dynamical variables $x_k$ (such as the canonical coordinate , momentum and spin ). Let $|n\rangle(n=1,2,...L)$ be states of the system (e.g., a quantum register) corresponding to energy levels $E_n$ $(n=1,2,...M)$. The system S with Hamiltonian

$$\hat{H}_s=\sum_{n=1}^{L}E_n|n\rangle\langle n| \tag{3.2}$$

interacts with the environment through the quantum non-demolition (QND)[43] interaction

$$\hat{H}_I=\sum_n\sum_j g_{n,j}(x_j)|n\rangle\langle n| \tag{3.3}$$

In a special realization for quantum computation, $|n\rangle$ may denote an array of qubits, each of which has two states $|0\rangle$ and $|1\rangle$, through the definition $|n\rangle=|n_o,n_1,n_2,...,n_{L-1}\rangle$ where the labels satisfy the unique binary representation

$$n=\sum_{i=0}^{L-1}n_i2^i(n_i=0,1). \tag{3.4}$$

It should be emphasized that, for quantum measurement, the interaction between S and E must be chosen to have the different strengths for the different states of S, i.e., it is required that $g_{nj}\neq g_{mj}$ for $m\neq n$. In fact, the so-called measurement is a scheme to read out the states of S from the counting number of detector such that different counting numbers should correspond to different states of S. However, this requirement of non-degeneracy is not necessary when we extend this generalized HC

model such that it is applicable in quantum computation.

If the coupling of the system to the environment is degenerate, namely, $g_{n,j}(x_j) = g_{m,j}(x_{j)}$ for certain $n \neq m$, we can group the coefficients of the interaction as follows

$$g_{1,j} = .... = g_{d_1,j} \equiv \kappa_{1,j},$$

$$g_{d_1+1j} = .... = g_{d_1+d_2j} \equiv \kappa_{2,j},$$

$$.....$$

$$g_{d_1+..+d_{q-1}+1j} = .... = g_{d_1+...+d_qj} \equiv \kappa_{q,j}$$

$$..... \quad (3.5)$$

Correspondingly, the Hilbert space $V : \{|n\rangle | n = 1, 2, ...L\}$of S is decomposed into a direct sum

$$V_s = \sum_q \oplus V^q$$

of the subspaces

$$V^1 : \{|n = m\rangle \equiv |1, m\rangle | m = 1, ...d_1\}$$

$$V^2 : \{|n = m + d_1\rangle \equiv |2, m\rangle | m = 1, ...d_2\}$$

$$.....$$

$$V^q : \{|n = d_1 + .. + d_{q-1} + m\rangle \equiv |q, m\rangle | m = 1, ..., d_q\}$$

$$..... \quad (3.6)$$

Notice that $\kappa_{qj} \neq \kappa_{q'j}$ for $q \neq q'$. With the above decomposition of the Hilbert space, the interaction Hamiltonian can be re- written as

$$H_I = \sum_{q,m} \sum_j \kappa_{q,j}(x_j)|q, m\rangle\langle q, m| \quad (3.7)$$

We observe that the coupling has the same strength for the states belonging to the same subspace $V^q$.

In some cases the above classification of state vectors is a reflection of the structure of some irreducible representation of certain group chain $G \supset K$ where $G, K$ are such chosen that $H_I$ is $G-$invariant and $H_s$ is at most $K-$ invariant. For instance, consider the group chain $SO(3) \supset SO(2)$ . It defines the standard angular basis$|J, M\rangle$ through the Casimir operators $\hat{J}^2$ and $\hat{J}_3$ of $SO(3)$ and $SO(2)$. In this case, we can take $\hat{H}_s = H(\hat{J}^2, \hat{J}_3)$ to be the Zeeman Hamiltonian in a central force field (not Coulomb field) if the interaction $H_I$ $=H_I(\hat{J}^2)$.

A special cases of the above general discussion has already been given in ref. [21]. They introduce a totally factorized interaction as of form $H_I = \hat{Q} \otimes \sum_{j=1}^N f_j(x_j)$. Here $\hat{Q}$ is a system variable commuting with the free Hamiltonian $\hat{H}_s$ of the qubit system and $x_j$ (j=1,2,....N) are the variables of the environment with the free Hamiltonian $\hat{H}_D = \sum_k^N \hat{H}_k$. The Hilbert space $V_s$ for the system can be spanned by $|q, m\rangle (m = 1, 2, ..., d_q$ for a given $q$ ) , the common eigenstates of $\hat{Q}$ and $\hat{H}_s$ labeled by $q$ and $m$.

$$\hat{Q}|q, m\rangle = e_q|q, m\rangle, \hat{H}_s|q, m\rangle = E_{qm}|q, m\rangle \quad (3.8)$$

In this case we have the direct sum decomposition $V_s = \sum_q \oplus V^q$ with the eigen-spaces $V^q = Span\{|q, m\rangle | m = 1, 2, ..., d_q\}$.This special interaction can be extended to a most general form with many system variables $\hat{Q}_j (j = 1, 2..K)$ that cancel certain subspaces of the qubit system simultaneously [21 ]. In the view point of group representation theory, this generalization enjoys an elegant mathematical structure [22, 23].

## 4. State Reduction in Time Evolution

In this section it is shown that the above general structure of subspace decomposition (3.5) indeed dynamically leads to a scheme grouping the states of qubit system to avoid decoherence. Let $V_d = V_1 \otimes V_2 \otimes .... \otimes V_{N-1} \otimes V_N$ denote the direct product Hilbert space for the environment. Here $V_k (k = 1, 2, ..., N)$ denotes the Hilbert space of the k'th particle comprising the environment. We will prove that, with the QN interaction (3.6) any coherent superposition $\sum_m C_m|q, m\rangle$ of the states belonging to the same subspace $V^q$ is decoherence free while the coherent superposition $\sum_q D_q|q, m_q\rangle$of the states belonging to different subspaces may experience a WFC or decoherence. For a given initial state $|\sigma(0)\rangle = |\sigma_j(0)\rangle \in V_d$, of the environment, and a given initial state $|f(0)\rangle = \sum_{m,q} C_m^q|q, m\rangle$, of the system , the general initial state of the total system $|\Phi(0)\rangle = |f(0)\rangle \otimes|\sigma(0)\rangle$ will evolve into an entangling state

$$|\Phi(t)\rangle = \sum_q \sum_m^{d_q} \exp[-iE_{qm}t]C_m^q|q, m\rangle\}$$

$$\otimes \prod_j \exp\{-it[\hat{H}_k + \kappa_{q,j}(x_j)]\}|\sigma_j(0)\rangle \quad (4.1)$$

where $E_{qm} = E_{d_1+..+d_{q-1}+m}, m = 1, 2, ..d_q$. Notice that the states belonging to a subspace $V^q$ entangle with the environment through the same factorized components

$$\prod_j U_j^q(t)|\sigma_j(0)\rangle \equiv$$

$$\prod_j \exp\{-i[\hat{H}_k + \kappa_{q,j}(x_j)]t\}|\sigma_j(0)\rangle \tag{4.2}$$

This kind of factorization structure in evolution of wave function is crucial to decoherence or WFC [30-36]. In fact, the reduced density matrix of the system at time t is

$$\rho(t) = Tr_d(|\Phi(t)\rangle\langle\Phi(t)|) =$$

$$\sum_q \{\sum_m |C_m^q|^2 |q,m\rangle\langle q,m| +$$

$$\sum_{m\neq .m'} \exp[iE_{qm'}t - iE_{qm}t] C_m^q C_{m'}^{q*} |q,m\rangle\langle q,m'|\}$$

$$+ \sum_{q\neq q'} \sum_{m.m'} \exp[iE_{q'm'}t - iE_{qm}t] C_m^q C_{m'}^{q'*} |q,m\rangle$$

$$\langle q',m'| \prod_{j=1}^N \langle\sigma_j(0)|U_j^{q'\dagger}(t)U_j^q(t)|\sigma_j(0)\rangle \tag{4.3}$$

where $Tr_d$ means taking partial trace over the variables of the environment. From this expression we see that each off-diagonal element of $\rho(t)$ , labeled by $q$ and $q'$, is accompanied by a decohering factor

$$F_{q,q'}(N,t) = \prod_{j=1}^N \langle\sigma_j(0)|U_j^{q'\dagger}(t)U_j^q(t)|\sigma_j(0)\rangle$$

$$\equiv \prod_{j=1}^N F_{q,q'}^j(t) \tag{4.4}$$

in the form of factorized function .Obviously, if the initial state $|f(0)\rangle$ belongs to a single subspace $V^q$, then the terms accompanied by $F_{q,q'}(N,t)$ do not appear. Thus the system will remain in the pure state $\exp[-iH_st]|f(0)\rangle\langle f(0)|\exp[iH_st]$ throughout the evolution process. This fact is significant for developing schemes to carry out error free quantum computations. The expression also manifests the happening of decoherence for those states belonging to different subspaces $\{V^q\}$.

Next we consider the dynamic process of decoherence when a superposition of states mixes the vectors belonging to different subspaces. Naively, as $F_{q,q'}(N,t)$ is a multiplication of $N$ factors $F_{q,q'}^j(t)$ with norms not larger than the unity, it may approach zero in the macroscopic limit with very large $N$. To deal with this problem precisely, we define an real number not less than zero

$$\Delta_j^{q,q'}(t) = -\ln|F_{q,q'}^j(t)| \tag{4.5}$$

Then the norm of the accompanying factor $F_{q,q'}(N,t)$ is expressed as

$$|F_{q,q'}(N,t)| = \exp[-\sum_{j=1}^N \Delta_j^{q,q'}] \tag{4.6}$$

Obviously, the series $\sum_{j=1}^\infty \Delta_j^{q,q'}(t) \geq 0$ since each term is not less than zero. There are two cases in which the accompanying factor $F_{q,q'}(N,t)$ approaches zero in the macroscopic limit with very large $N$. The first case is that the series $\sum_{j=1}^\infty \Delta_j^{q,q'}(t)$ diverges on $(0,\infty]$. The second case is that the series converges to a monotonic function which approaches zero as $N\to\infty$. Therefore, it is possible that

$$\rho(t) \to \sum_q \{\sum_m |C_m^q|^2 |q,m\rangle\langle q,m| +$$

$$\sum_{m\neq .m'} \exp[iE_{qm'}t - iE_{qm}t] C_m^q C_{m'}^{q*} |q,m\rangle\langle q,m'|\} \tag{4.7}$$

as $N\to\infty$. In the next section, some examples will be presented to illustrate the above mentioned circumstances explicitly.

## 5.Dynamic Decoherence in Environment Consisting of N Two- Level Subsystems

To make a deeper elucidation of the above mentioned quantum dynamic mechanism of decoherence in quantum computation and the relevant scheme of grouping the states of a qubit system to be prevented from decoherence, in this section, we model the environment as consisting of $N$ two level subsystems. We recall that Caldeira and Leggett [37] have pointed out that any environment weakly coupling to system may be approximated as a bath of oscillators . On the condition that "any one environmental degree of freedom is only weakly perturbed by its interaction with the system", they have also justified describing the influence of environment by a coupling linear in the bath variables up to the first order perturbation. With this justification, we observed that any linear coupling only involves the transitions between the lowest two levels (ground state and the first excitation state) of each harmonic oscillator in environment though it has many energy levels. Therefore in this case we can also describe the environment as a combination of many two level subsystems without loosing generality. In fact, for quantum computation, Unruh[11] and

Palma et al [12] have considered the harmonic oscillator environment. Their model is equivalent to one introduced for the WFC in quantum measurement by Sun et.al [32, 31,3 4] . A similar model has also been touched by Leggett et al [37,38]and Gardiner[39] in studying the tunneling effect in a quantum dissipative process. Here we choose equivalently the two level subsystem model to manifest some characters independent of environment in the weakly coupling limit and to demonstrate explicitly the qualitative calculation of decoherence time through a sample example without quantum dissipation.

Let $|g_j\rangle$ and $|e_j\rangle$ be the ground and excited states of $j$ 'th subsystem .We define the quasi-spin operators

$$\sigma_1(j) = |e_j\rangle\langle g_j| + |g_j\rangle\langle e_j|$$

$$\sigma_2(j) = -i[|e_j\rangle\langle g_j| - |g_j\rangle\langle e_j|]$$

$$\sigma_3(j) = |e_j\rangle\langle e_j| - |g_j\rangle\langle g_j| \tag{5.1}$$

Then we introduce the Hamiltonian of the environment

$$H_e = \sum_{j=1}^{N} \hbar\omega_j \sigma_3(j) \tag{5.2}$$

and the interaction coupling to a qubit system

$$H_I = f(S)\sum_{j=1}^{N} \hbar g_j \sigma_2(j) \tag{5.3}$$

where $f(S)$ is function of the variable S of the qubit system.

In this section let us mainly focus on the simplest case where the system consists of two qubits with the Hamiltonian

$$H_s = \hbar\eta_1 S_3(1) + \hbar\eta_2 S_3(2) \tag{5.4}$$

where $S(1) = \sigma_s \otimes 1, S_s(2) = 1 \otimes \sigma_s, (s = 1, 2, 3)$ denote spin operators acting on the first and the second qubits respectively; $\sigma_s(s = 1, 2, 3)$ denoting the usual Pauli matrix. We consider the special interaction given by

$$f(S) = S_3(1) + S_3(2) \tag{5.5}$$

It means that in our model the interaction has the same strength for different states. This model is very simple, or even too simple in some sense. But we would like to point out that the so called Free Hamiltonian Elimination model in ref.[21] is substantially only a plain generalization of the present example to the multi-pair case if one takes into account the $SU(2)$ rotation transformation.

Let $|1\rangle$ and $|0\rangle$ be the qubit states that satisfy $S_3|k\rangle = (-)^{k+1}|k\rangle, (k = 1, 0)$. With the chosen interaction form, the Hilbert space, spanned by

$$\{|1,1\rangle = |1\rangle \otimes |1\rangle, |1,0\rangle = |1\rangle \otimes |0\rangle,$$
$$|0,1\rangle = |0\rangle \otimes |1\rangle, |0,0\rangle = |0\rangle \otimes |0\rangle\}$$

contains a null subspace $V^0$ of $H_I$ spanned by $|1,0\rangle$ and $|0,1\rangle$ . Any superposition $|\phi(0)\rangle = A|1,0\rangle + B|0,1\rangle$ in this subspace will preserve its purity in evolution process though the system has interaction with the environment. Precisely, the pure state $|\phi(0)\rangle\ \langle\phi(0)|$ will evolve into the pure state $U_0(t)|\phi(0)\rangle\ \langle\phi(0)|U_0^{\dagger}(t)$ where $U_0(t) = \exp[-i\eta_1 t S_3(1) - i\eta_2 t S_3(2)]$ is the free evolution operator of the qubit system. Physically, this fact implies that no useful information leaks out of the system in the process and the coherence is preserved. This analysis can be easily generalized to the many bit case where the free qubit Hamiltonian takes the form $H_s = \sum_{k=0}^{L-1} \hbar\eta_k S_3(k)$ and its interaction with the environment is determined by

$$f(S) = \sum_{k=0}^{L-1} \lambda_k S_3(k)] \tag{5.6}$$

where $L$ is the number of qubits used and $S_s(k) = \overbrace{1 \otimes \cdots \otimes 1}^{k-1 times} \otimes \sigma_s \otimes 1 \otimes \cdots \otimes 1$.The different $\lambda_k' s$ indicate that each single qubit has a different coupling to the same environment. In the Hilbert space of this L-qubit system with the basis

$$|q\rangle = |q_0\rangle \otimes ||q_1\rangle \otimes |q_2\rangle \otimes ... \otimes |q_{L-1}\rangle,$$

$$q_k = 0, 1; k = 0, 1, 2, ..., L \tag{5.7}$$

the subspace $V^{\xi}$ preserving coherence can be spanned by those basis vectors $|q\rangle$ satisfying

$$\sum_{k=0}^{L-1} \lambda_k (-)^{q_k+1} = const.\xi \tag{5.8}$$

Let us return to the two qubit example. If a superposition contains a vector outside the decoherence free subspace, decoherence will happen in an entanglement of system state with environment state. For example, if the initial state $|\varphi(0)\rangle = C|0,0\rangle + D|1,1\rangle$ of the system involves states not belonging to $V^0$ while the environment is initially in the vacuum state $|0\rangle_e = |g_1\rangle \otimes |g_2\rangle \otimes \cdots \otimes |g_N\rangle$ where $|g_j\rangle$is the ground state of $j'$th two level subsystem , the corresponding pure state density matrix $|\varphi(0)\rangle\ \langle\varphi(0)| \otimes |0\rangle_e \otimes {}_e\langle 0|$ of the total system formed by the qubits plus the environment will experience a unitary evolution to reach a pure state $\rho_T(t)$. Its reduced density matrix

$$\rho(t) = Tr_e \rho_T(t) = |C|^2 |0,0\rangle\langle 0,0| + |D|^2 |1,1\rangle\langle 1,1|$$

$$+[CD^* \exp[2i(\eta_1 + \eta_2)t] F(N,t) |0,0\rangle\langle 1,1| + H.c] \tag{5.9}$$

is no longer pure because the environment state becomes correlated with the system state. Here the decohering factor

$$F(N,t) \equiv \prod_{j=1}^{N} F_j(t) = \prod_{j=1}^{N} \langle g_j | U_{j1}^{\dagger}(t) U_{j0}(t) | g_j \rangle \quad (5.10)$$

is determined by the effective evolution operators

$$U_{j\alpha}(t) = \exp[-i\omega_j \sigma_3(j)t - i\xi_\alpha g_j \sigma_2(j)t]$$

$$\xi_1 = 2, \xi_0 = -2, (\alpha = 0, 1) \quad (5.11)$$

corresponding to the qubit states $|0,0\rangle$ and $|1,1\rangle$ respectively. Using the formula $\exp[i\overrightarrow{\sigma}\cdot\overrightarrow{A}] = \cos A + i\overrightarrow{\sigma}\cdot\overrightarrow{n_A}\sin A$ for a given vector $\overrightarrow{A}$ of norm $A$ along the direction $\overrightarrow{n_A}$, we get the explicit form of $U_{j\alpha}(t)$

$$U_{j\alpha} = \cos(\Omega_{j\alpha}t) - i\left[\sigma_2(j)\sin\theta_{j\alpha} + \sigma_3(j)\cos\theta_{j\alpha}\right]\sin(\Omega_{j\alpha}t) \quad (5.12)$$

where $\tan\theta_{j\alpha} = \frac{\xi_\alpha g_j}{\omega_j}$, $\Omega_{j\alpha} = \sqrt{(g_j\xi_\alpha)^2 + \omega_j^2}$. Then, we get the decohering factor $F(N,t) = \prod_{j=1}^{N} F(j,t)$, which is an $N-$multiple product of the factors $F(j,t) = 1 - 2\sin^2\theta_j \sin^2\Omega_j t$ of norm less than 1. Here, we have used the definitions $\tan\theta_j = \frac{2g_j}{\omega_j}$, $\Omega_j = \sqrt{4{g_j}^2 + \omega_j^2}$for the special labels $\xi_1 = 2, \xi_0 = -2$. Therefore, the temporal behavior of decoherence is described by

$$|F(N,t)| = \exp\sum_{j=1}^{N} \ln|1 - 8\frac{g_j^2}{\Omega_j^2}\sin^2(\Omega_j t)| \quad (5.13)$$

In the weakly coupling limit that $g_j \ll \omega_j$, we get

$$|F(N,t)| = e^{-S(t)} \equiv \exp\left(-\sum_{j=1}^{N}\frac{8g_j^2}{\omega_j^2}\sin^2(\omega_j t)\right) \quad (5.14)$$

A special case is that the subsystems constituting the environment are identical and the environment has a constant discrete spectrum, i.e., $\omega_k$ =constant $\omega$, $g_k$ =constant $g$. In this case, the off-diagonal elements with the factor $\exp\{-\frac{8Ng^2}{\omega^2}\sin^2\omega t\}$approach zero as $N \to \infty$ for all t except those satisfying $\omega t = 2k\pi (k = 0, 1, 2...)$. For general information, one needs a detailed analysis about the behavior of the series $S(t)$ for various spectrum distributions of the environment. Of special interest is the case with continuous spectrum. In such case $S(t)$ can be re- expressed in terms of a spectrum distribution $\rho(\omega_k)$ as

$$S(t) = \int_0^\infty \frac{8}{\omega_k^2}\rho(\omega_k)g_k^2\sin^2\omega_k d\omega_k \quad (5.15)$$

Notice that, in the case of discrete spectrum, the distribution means a degeneracy : there are $\rho(\omega_k)$ subsystems possessing the same frequency $\omega_k$. From some concrete spectrum distributions,interesting circumstances may arise. For instance, when $\rho(\omega_k) = \frac{1}{\pi}\gamma/g_k^2$ the integral converges to a negative number proportional to time t , namely, $S(t) = -\gamma t$. This shows that the norm of the decoherence factor is exponentially decaying and as $t \to \infty$, the off-diagonal elements of the density matrix vanish simultaneously! Another example of continuous spectrum is the Ohmic type [37,38] $\rho(\omega_k) = \frac{2\eta\omega_k^2}{\pi g_k^2}$, which leads to a diverging integral $S(t) \to \infty$ for $t \neq 0$. In conclusion, in the present example, we can choose a suitable spectrum distribution of the oscillators in the detector, such that the series $S(t)$ diverges to infinity, or in other words, the dynamical evolution of the system plus environment results in the complete decoherence in the reduced density matrix of S independent of the temperature. However, it is only an accident situation owing to the special choice of the initial state. For a general initial state, we will see, the decoherence process indeed shows a temperature independence.

## 6. Decoherence Time for L-Qubit System

Usually, when the coherence of a quantum system develops a characteristic decay proportional to a factor of the form $\exp(-t/t_d)$, $t_d$, which characterizes the speed of the decoherence or the transition of the system from the quantum regime to the classical one, is called the decoherence time. Its value depends on the physical feature of the quantum system and their interaction with the environment. For a single qubit system some numerical estimates of $t_d$ have been made by DiVincenzo[44] for several physical realizations. It ranges from $10^4$ s (for nuclear spins )to $10^{-12}$ s (for the electron-hole excitation in bulk of a semiconductor). In practice, to carry out a quantum computation, one needs a large number of qubits, e.g., in Shor's algorithm factoring large number $n$, $L \propto \ln n$. Accordingly, in the following we extend the dynamic analysis to show how the speed of decoherence becomes larger as the number of qubits increases.

Let us consider the L-bit system coupling to the environment mentioned in the last section. The interaction constants $\lambda_k$'s are chosen so that eigen-values

$$\xi(q) = \sum_{k=0}^{L-1}\lambda_k(-1)^{q_k+1} \quad (6.1)$$

are not degenerate for $\{q_k = 0, 1\}$.Starting from an initial state $|\varphi(0)\rangle = C|q\rangle + D|q'\rangle$, where

$$|p\rangle = \prod_{k=1}^{L-1}\otimes|p_k\rangle, p = q, q' \quad (6.2),$$

the initial pure state density matrix of the total system formed by the qubits plus the environment will experience a unitary evolution to reach a pure state $\rho_L(t)$.

Imitating the calculation process in the last section, we can obtain the reduced density matrix $\rho(t) = Tr_e\rho_L(t)$. Its off-diagonal elements are proportional to the decohering factor

$$F_L(N,t) = \prod_{j=1}^{N} F_L(j,t)$$

$$\equiv \prod_{j=1}^{N} \langle g_j|U_{jq}^{\dagger}(L,t)U_{jq'}(L,t)|g_j\rangle \tag{6.3}$$

where

$$U_{jq}(L,t) \equiv \prod_{j=1}^{N} \exp[-i\omega_j\sigma_3(j)t - i\xi(q)g_j\sigma_2(j)t] \tag{6.4}$$

Using the notions

$$\tan\theta_j(q) = \frac{\xi(q)g_j}{\omega_j}, \quad \Omega_j(q) = \sqrt{[g_j\xi(q)]^2 + \omega_j^2} \tag{6.5}$$

and the matrix representation of $U_{jq}(L,t)$, after straight calculation we get

$$F_L(j,t) = \sin\theta_j(q)\sin[\Omega_j(q)t]\sin\theta_j(q')\sin[\Omega_j(q')t]$$

$$+\{\cos[\Omega_j(q)t] - i\cos\theta_j(q)\sin[\Omega_j(q)t]\}$$

$$\times\{\cos[\Omega_j(q')t] + i\cos\theta_j(q')\sin[\Omega_j(q')t]\} \tag{6.6}$$

Trivially, $F_L(N,t)$ becomes unity when $q = q'$. However, when $q \neq q'$, in the weakly-coupling limit $g_j \ll \omega_j$, we have

$$\sin\theta_j(q) \simeq \theta_j(q)$$

$$\cos\theta_j(q) \simeq 1 - \frac{1}{2}\theta_j^2(q)$$

$$\Omega_j(q) \simeq \omega_j \tag{6.7}$$

Thus $F_L(j,t) \simeq 1 - \frac{1}{2}\{\theta_j(q) - \theta_j(q')\}^2\sin^2(\omega_j t) + \frac{i}{4}\{\theta_j^2(q) - \theta_j^2(q')\}\sin(2\omega_j t)$. Since in such weak-coupling limit $\theta_j(q) \simeq \sin\theta_j(q) \simeq \xi(q)\frac{g_j}{\omega_j}$, we obtain the decohering factors

$$F_L(j,t) \simeq 1 - \frac{g_j^2}{2\omega_j^2}\{\xi(q) - \xi_j(q')\}^2\sin^2(\omega_j t)$$

$$+\frac{ig_j^2}{4\omega_j^2}\{\xi_j^2(q) - \xi_j^2(q')\}\sin(2\omega_j t) \tag{6.8}$$

Consequently

$$|F_L(N,t)| = \exp\{-[\xi(q) - \xi(q')]^2\sum_{j=1}^{N}\frac{g_j^2}{2{\omega_j}^2}\sin^2(\omega_j t)\} \tag{6.9}$$

In summary, the temporal behavior of the decoherence is described by $F(N,t)$, and actually determined by $|F(N,t)|$ , which is of the form $\exp[-S_L(t)]$. Here $S_L(t) = [\xi(q) - \xi(q')]^2\sum_{j=1}^{N}\frac{g_j^2}{2{\omega_j}^2}\sin^2(\omega_j t)$. For identical qubits $\lambda_k$ =1,the fastest decoherence happens between the two initial states $|q\rangle = |q_0 = 1\rangle \otimes |q_1 = 1\rangle \otimes \cdots \otimes |q_{L-1} = 1\rangle$ and $|q'\rangle = |q_0 = 0\rangle \otimes |q_1 = 0\rangle \otimes \cdots \otimes |q_{L-1} = 0\rangle$ . In this case $|F_L(N,t)| = \exp[-L^2S(t)]$. Thus for the instance with $S(t) = \gamma t$, which is discussed in the last section, we have $|F(N,t)| = \exp[-L^2\gamma t]$ where $\gamma^{-1}$is the decoherence time for a single qubit. This shows that the characterized time of the fastest decoherence happening in the L-qubit system is $L^2$ times of that of a single qubit. This conclusion first obtained by Palma et.al.[12] is given here in the framework of quantum dynamic model of decoherence.

## 7.Temperature Dependence of Decoherence

The above discussion about decoherence in quantum computation only concerns the situation of zero temperature. In this section we consider the influence of environment at a finite temperature. Suppose the initial state of the total system is described by a density matrix $\rho\,(0) = \rho_s(0)\otimes\rho_b(0)$ where $\rho_s(0) = |\phi(0)\rangle\langle\phi(0)|$ is the density matrix of the system, while $\rho_b(0)$ is that of the bath

$$\rho_b(0) = \frac{\exp(-\beta\hat{H}_b)}{Tr_b\exp(-\beta\hat{H}_b)} = \prod_{j=1}^{N}\rho_{jb}(0)$$

$$= \prod_{j=1}^{N}\frac{e^{-\beta\omega\sigma_3(j)}}{2\cosh(\beta\omega_j)} \tag{7.1}$$

with $\beta = 1/(K_BT)$. For the initial state $|\phi(0)\rangle = A|0,0\rangle + B|1,1\rangle$, we obtain the same decohering factor $F_2(N,t) = \prod_j^N[1 - 2\sin^2\theta_j\sin^2(\Omega_j t)]$ which is independent of temperature. This is due to the special choice of the initial state with a certain permutation symmetry between $|1,0\rangle$ and $|0,1\rangle$. For a general initial state $|\varphi(0)\rangle = C|q\rangle + D|q'\rangle$, we can calculate the factor $F_L(N,t) = \prod_{j=1}^{N} F_L(j,t)$ as follows

$$F_L(j,t) \equiv Tr_b[U_{jq'}(L,t)\rho_{jb}(0)U_{jq}^{\dagger}(L,t)]$$

$$= \sin\theta_j(q)\sin[\Omega_j(q)t]\sin\theta_j(q')\sin[\Omega_j(q')t] +$$

$$\cos[\Omega_j(q)t]\cos[\Omega_j(q')t]+$$

$$\cos\theta_j(q)\sin[\Omega_j(q)t]\cos\theta_j(q')\sin(\Omega_j(q')t)$$

$$-\frac{i}{2}\tanh(\beta\omega_j)\sin(2\Omega_j(q)t)\{\cos\theta_j(q)-\cos\theta_j(q')\} \quad (7.2)$$

Notice that the effect of finite temperature only appears in the imaginary part of the decohering factor. In the weakly coupling limit, it is not difficult to observe that $F_L(j,t)\simeq 1-\frac{1}{2}\{\theta_j(q)-\theta_j(q')\}^2\sin^2[\omega_j t] - \frac{i}{4}\tanh(\beta\omega_j)\sin[2\omega_j t][\theta_j^2(q)-\theta_j^2(q')]$ or

$$F_L(j,t)\simeq 1-\frac{g_j^2}{2\omega_j^2}\{\xi(q)-\xi_j(q')\}^2\sin^2(\omega_j t)$$

$$+\frac{ig_j^2}{4\omega_j^2}\{\xi_j^2(q)-\xi_j^2(q')\}\tanh(\beta\omega_j)\sin(2\omega_j t) \quad (7.3)$$

It reflects the novel fact that, in an environment weakly interacting with the qubit system, the decoherence time do not depends on temperature as a result of the temperature-independent norm of $F_L(j,t)$. Thus in this case thermal fluctuation plays a role in quantum computation only through affecting the phases of the off-diagonal elements of the reduced density matrix.

## 8. Universality of Environment in Weak Coupling Limit

An environment surrounding a qubit system for quantum computation maybe very complicated. Intuitively, the dynamic process of decoherence in quantum computation should depend on the details of interaction between the qubit system and the environment. So generally it seems impossible to control decoherence in a qubit system. Nevertheless, one may well expect that in some limit situations there exists certain universality in the dynamics of interaction so that the physical parameters (such as the decoherence time and decoherence factor) dominating a quantum computation process would not depend on the detail of environment . For the tunneling problem in quantum dissipation process, this kind of universality has been considered by Caldeira and Leggett [37, 38] by modeling the environment as a bath of harmonic oscillators with a linear coupling to the system. In this section, we illustrate that, in the weakly coupling limit, the above results obtained from the two-level subsystem model of environment coincide with those from the harmonic oscillator model concerned in various quantum irreversible processes, such as wave function collapse [32,34,31] and quantum dissipation [37-42].

Let $a_i^+$and $a_i$ be the creation and annihilation operators for the i'th harmonic oscillator in the environment. The Hamiltonian of the environment takes the form $H=\sum_{j=1}^N\hbar\omega_j a_j^+ a$ and its interaction with the qubit system can be modeled as a linear coupling:

$$H_I=f(s)\sum_{j=1}^N\hbar g_j(a_j^++a_j) \quad (8.1)$$

where $f(s)$ is a linear or non-linear function of the qubit system variable $s$. Let the initial state of the qubit system $|\varphi(0)\rangle=A|\alpha\rangle+B|\beta\rangle$ be a coherent superposition of two eigenstates of $s, s|\alpha\rangle=\alpha|\alpha\rangle, s|\beta\rangle=\beta|\beta\rangle$ and let the environment be initially in the vacuum state $|0\rangle_e=|0_1\rangle\otimes|0_2\rangle\otimes\cdots\otimes|0_N\rangle$ where $|0_j\rangle$is the ground state of the j'th single harmonic oscillator. The corresponding decohering factor $F(N,t)=\prod_{j=1}^N {}_h\langle 0|U_j^{\beta\dagger}(t)U_j^{\alpha}(t)|0\rangle_h\equiv\prod_{j=1}^N F_j(t)$can be obtained by solving the Schrodinger equations of $U_j^{\gamma}(t)$ $(\gamma=\alpha.\beta)$ governed by the Hamiltonian of forced harmonic oscillator

$$H_{j\gamma}=\hbar\omega_j a_j^+a_j+f(\gamma)g_j(a_j^++a_j) \quad (8.2)$$

In fact, by the so called Wei-Norman algebraic expansion technique one has the following explicit result [12,32,35].

$$F(N,t)=\exp\{-[f(\alpha)-f(\beta)]^2\sum_{j=1}^N\frac{2g_j^2}{{\omega_j}^2}\sin^2(\frac{\omega_j t}{2})\}$$

$$\times\exp\{-i[f(\alpha)^2-f(\beta)^2]\sum_{j=1}^N\frac{g_j^2}{\omega_j}[t+\frac{\sin(\omega_j t)}{\omega_j}]\} \quad (8.3)$$

The decoherence time is decided by the real part of $F(N,t)$, which is the same as in eq.(6.8) from the two level model of environment in the weakly coupling limit. This can easily be seen if only one replaces $\frac{\omega_j}{2}$ in the above equation by $\omega_j$. Thus in the weakly coupling limit the differences among different models of environment are only reflected in the imaginary parts of the decohering factors This simply implies that in this case the details of environment does not affect the speed at which a quantum system approaches the classical kingdom. But they do affect the success probability of a quantum computation.

## 9. A Discussion

We have seen from our model that for a quantum register with L qubits the relevant coherence develops a characteristic decay proportional to $\exp[-L^2/t_d t]$ where $t_d$ is the typical decoherence time for a single bit. Thus if a quantum algorithm calls for K elementary computation steps and each step takes time $\tau$ on the average in order that the algorithm could be feasible we should have the condition

$$L^2\tau K<t_d \quad (9.1)$$

Generally speaking, this would pose a strong restriction on L and K. We need to develop proper quantum error correction schemes to cope with this difficulty caused by decoherence , which is unavoidable in the quantum kingdom. Along this line there has been some progress. Nevertheless there is another severe problem which may endanger the assumed great utility of quantum computers. In section 2 we have shown that environment may affect the efficiency of a quantum algorithm. Although our discussion is not sophisticated enough it indeed gives us a frustrating information. This problem deeply rooted in the quantum kingdom seems to have been ignored. We think it is now time to face it seriously.

*This work is supported in part by special project of the NSF of China and the National Excellence Youth Foundation of China*

## References


1 D. Deutsch, Proc. R. Soc. Lond. A **400**, 97 (1985).

2 S. LIoyd, Science **261**, 1569 (1993).

3 A. Barenco, Contemp. Phys. **37**, 375 (1996).

4 C. H. Bennett, Phys. Today , **47(10)**, 24 (1995).

5 A. Ekert and R. Jozsa, Rev. Mod. Phys. 68, 733 (199G)

6 P. Shor, i*n Proceedings of the 35th Annual Symposium on Foundations of Computer Science 1994* (IEEE Computer Society Press, Los Alamitos, C.A., 1994), p.l24-134.

7 J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).

8 Q. A. Turchette, C. J. Hood, W. Lange, H. Mabuchi and H. J. Kimble Phys. Rev.Lett. **75**, 4710 (1995);

9 C. Monroe, D. M. Meekhof, B. E. King, W. M. Itano and D. J. Wineland, Phys. Rev. Lett. **75**, 4714 (1995).

10 I. L. Chuang, R. Laflamme, P. W. Shor and W. H. Zurek, Science **270**, 1633 (1995).

11 W. G. Unruh, Phys. Rev. A **51**, 992 (1995).

12 G. M. Palma, K. A. Suominen, and A. K. Ekert, Proc. R. Soc. London, A **452**, 567 (1996).

13 W.H.Zurek, Phys.Today, **44(10)**, 36 (1991).

14 P. W. Shor, Phys. Rev. A **52**, R2493 (1995).

15 A. M. Steane, Phys. Rev. Lett. **77**, 793 (1996).

16 A. R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098 (1996).

17 R. Laflamme, C. Miguel, J. P. Paz, and W. H. Zurek, Phys. Rev. Lett. **77**, 198 (1996).

18 C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).

19 P. W. Shor and R. Laflamme, Phys. Rev. Lett. **78**, 1600 (1997).

20 H. F. Chau, Phys. Rev. A55, R839 (1997).

21 L.M. Duan and G.c.Guo,Phys. Rev. Lett. **79**, 1953 (1997).

22 P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79**, 3306(1997).

1. P. Zanardi and M. Rasetti, Lett.Math.Phys., in press (1998)

24 J.von Neumann, *Mathemstische Gruandlage de Quantumechanik*, (Berlin, Julius, 1932).

25 K. Hepp, Hev.Phys.Acta, **45**, 237 (1972).

26 J.S. Bell, Hev.Phys.Acta, **48**, 93 (1975).

27 M. Cini, Nuovo Cimento,**73B**, 27(1983).

28 M. Namik and S.Pascazio, Phys.Rev., A**44**, 39(1991).

29 H. Nakazato and S. Pascazo, Phys.Rev.Lett., **70**, (1993).

30 C.P.Sun, Phys.Rev.A, **48**, 878 (1993).

31 C.P.Sun, Chin. J. Phys., **32**, 7(1994)

32 C.P. Sun, X.X. Yi, and X.J. Liu, Fortschr.Phys., **43**, 585 (1995)

33 X.J.Liu and C.P.Sun, Phys.Lett.A, **198**, 371(1995)

34 C.P.Sun, X.X.Yi, S.Y.Zhao, L.Zhang and C.Wang, Quantum Semiclass.Opt. **9**,119(1997)

35 C.P. Sun, *General Dynamical Model of Quantum Measurement and Its Application to quantum Zeno Effect,* in Proceedings of the 4th Drexel Symposium On Quantum noninterability, Philadelphia 1994, (International Press, 1996).

36 C.P. Sun,*Generalized Hepp-Coleman Models for Quantum Decoherence as a Quantum Dynamic Process*, Quantum Coherence and Decoherence, ed.by K.Fujikawa and Y.A.Ono,, pp.331-334,(AMsterdam: Elsevier Sciense Press,1996)

37 A. O. Caldeira and A. J. Leggett, *Ann. Phys.* (N.Y.), **149,** 374(1983).

38 A.J. Leggett, S. Chakravarty, A.T. Dosey, M.P.A .Fisher and W. Zwerger, Rev.Mod.Phys, **59**, 1-87(1987).

39 C. Gardiner, *Quantum Noise*, (Berlin, Springer,1991).

40 L.H.Yu and C.P. Sun, Phys.Rev.A, **49,** 592(19940).

41 C.P.Sun and L.H. Yu, Phys.Rev.A, **51**,1845(1995).

42 C.P.Sun, Y.B.Gao, H.F. Dong and S.R.Zhao, Phys.Rev.E. **57,** in press, (1998)

43 V. Braginsky and Khalili, *Quantum Measurement*, ( Cambridge Univ. Press, London 1992).

44 D.P. DiVincenzo, Science, **270**, 255(1995)